A FEW REMARKS ON SAMPLING OF SIGNALS WITH SMALL SPECTRUM

SHAHAF NITZAN, ALEXANDER OLEVSKII AND ALEXANDER ULANOVSKII

ABSTRACT. Given a set S of small measure, we discuss existence of sampling sequences for the Paley-Wiener space PW_S, which have both densities and sampling bounds close to the optimal ones.

1. INTRODUCTION

Let $S \subset [0, 2\pi]$ be a set of positive measure. Denote by PW_S the space of all functions $f \in L^2(\mathbb{R})$ whose Fourier transform,

$$\hat{f}(t) := \int_{\mathbb{R}} e^{-ixt} f(x) \, \frac{dx}{2\pi},$$

is supported by S, endowed with the L^2-norm.

A sequence $\Lambda \subset \mathbb{Z}$ is called a sampling sequence for PW_S if there exists $B > 0$ such that

$$B\|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \qquad \forall f \in PW_S. \tag{1}$$

Landau's theorem [7] states that sampling sequences cannot be too sparse compared to the measure of S: If Λ is a sampling sequence for PW_S, then

$$D^-(\Lambda) := \lim_{r \to \infty} \frac{\min_{x \in \mathbb{R}} |\Lambda \cap [x, x + r]|}{r} \geq |S|,$$

where $|\cdot|$ denotes the normalized Lebesgue measure on $[0, 2\pi]$.

In this note we discuss the existence of "good" sampling sequences $\Lambda \subset \mathbb{Z}$. The following observation explains what we mean by that: Assume $S = [0, 2\pi/m]$, where $m \in \mathbb{N}$. For every $j \in \mathbb{N}$, the sequence $\Lambda = \{j + mn : n \in \mathbb{Z}\}$ is an excellent sampling sequence for PW_S: It is uniformly distributed in $\mathbb{Z}$, its density is equal to $|S|$ (the minimal density allowed, by Landau's theorem) and

$$|S|\|f\|^2 = \sum_{\lambda \in \Lambda} |f(\lambda)|^2. \tag{2}$$

For arbitrary sets S, we show the existence of sampling sequences Λ with both densities and bounds close to the ones described in this canonical example. When S has a "large" measure, the existence of such sequences follows from Bourgain and Tzafriri's Restricted Invertibility Theorem [2] (see also [14]). However, this result apparently is not applicable in the case when the spectrum S has a "small" measure

(see Remark 1). Our results are based instead on a recent finite-dimensional theorem by Batson, Spielman and Srivastava [1].

Observe that in [11] Olevskii and Ulanovskii constructed universal sampling sequences with density near optimal, which provide sampling for all compacts of a given measure. For another approach to this result see [10]. However, it was proved in [11] that such sampling sequences never have "universally good" bounds.

We distinguish two cases: when S is a compact, and when it is a general (measurable) set. For a compact set S there exists a good sampling sequence which is uniformly distributed in $\mathbb{Z}$. To be more precise, set

$$D^+(\Lambda) := \lim_{r \to \infty} \frac{\max_{x \in \mathbb{R}} |\Lambda \cap [x, x+r]|}{r}.$$

If $D^-(\Lambda) = D^+(\Lambda)$, one says that Λ has uniform density $D(\Lambda) := D^-(\Lambda) = D^+(\Lambda)$.

Theorem 1. *Fix $d > 0$. Given a compact set $S \subset [0, 2\pi]$, there exists $\Lambda \subset \mathbb{Z}$ with $D(\Lambda) < (1+d)|S|$ such that*

$$C(d)|S|\|f\|^2 \leq \sum |f(\lambda)|^2 \qquad \forall f \in PW_S,$$

where $C(d)$ is a positive constant depending only on d.

Moreover, in this case the sequence Λ can be chosen as a finite union of arithmetic progressions.

For general sets S the situation is more delicate. In this case, we do not know whether sequences Λ of uniform density can provide good sampling sequences. However, if the uniform density is replaced by a weaker notion of density:

$$D^\sharp(\Lambda) := \lim_{n \to \infty} \frac{\Lambda \cap [-n, n]}{2n},$$

(provided the limit exists), then the result holds.

Theorem 2. *Fix $d > 0$. Given a set $S \subset [0, 2\pi]$ of positive measure, there exists $\Lambda \subset \mathbb{Z}$ with $D^\sharp(\Lambda) < (1+d)|S|$ such that*

$$C(d)|S|\|f\|^2 \leq \sum |f(\lambda)|^2 \qquad \forall f \in PW_S,$$

where $C(d)$ is a positive constant depending only on d.

As mentioned above, these theorems are based on a result in [1], which studied the existence of small, "well invertible", submatrices. This result is presented in Section 2. Theorem 1 is then proved in Section 3. In Section 4 we use Theorem 1 and a Theorem of Ruzsa [12] to obtain Theorem 2. In Section 5 we discuss the upper bounds and in Section 6 some other related questions and open problems.

Set $\mathbb{T} := [0, 2\pi]$. Throughout the rest of this paper we assume that $\Lambda \subset \mathbb{Z}$ and $S \subset \mathbb{T}$. We denote by $|S|$ the normalized Lebesgue measure of a set S, and by $|J|$ the number of elements in a finite set J. We let l_2^n be the n dimensional space over $\mathbb{C}$ with norm $\|\{a_r\}_{r=1}^n\|^2 = \sum_{r=1}^n |a_r|^2$. For $J \subset \{1, ..., n\}$ we denote by $l_2(J)$ the $|J|$ dimensional subspace of l_2^n, indexed by J. Given a matrix A of order $m \times n$,

and a subset $J \subseteq \{1, ..., m\}$, we denote by $A(J)$ the sub-matrix of A with rows belonging to the index set J.

2. Batson, Spielman and Srivastava's theorem

Theorem A. *Let $\{v_i\}_{i=1}^m$ be a system of vectors in l_2^n, $n \leq m$, which satisfies*

$$\sum_{i=1}^m |\langle w, v_i \rangle|^2 = \|w\|^2 \qquad \forall w \in l_2^n. \tag{3}$$

Then for every $d > 0$ there exist, a subset $J \subset \{1, ..., m\}$ and positive weights $\{s_i\}_{i \in J}$, such that $|J| \leq (1 + d)n$ and

$$A(d)\|w\|^2 \leq \sum_{i \in J} s_i |\langle w, v_i \rangle|^2 \leq B(d)\|w\|^2 \qquad \forall w \in l_2^n,$$

where $A(d)$ and $B(d)$ are positive constants depending only on d.

Observe, that this theorem is formulated in [1] for the real spaces. However, the proof works also for the complex spaces.

Assume additionally that $\|v_i\|^2 = n/m$ for every $i = 1, ..., m$. Then, by putting $w = v_j$, $j \in J$, we find

$$s_j |\langle v_j, v_j \rangle|^2 \leq \sum_{i \in J} s_i |\langle v_j, v_i \rangle|^2 \leq B(d)\|v_j\|^2,$$

which implies that $s_j \|v_j\|^2 \leq B(d)$ or $s_j \leq B(d)m/n$ for every j. It follows that

$$C(d)\frac{n}{m}\|w\|^2 \leq \sum_{i \in J} |\langle w, v_i \rangle|^2 \qquad \forall w \in l_2^n$$

where $C(d)$ is a positive constant depending only on d. Hence, we get

Corollary 1. *Let A be an $m \times n$ matrix which is a sub-matrix of some $m \times m$ orthonormal matrix, and such that all of it's rows have equal l^2 norm. Then for every $d > 0$ there exists a subset $J \subset \{1, ..., m\}$ for which $|J| \leq (1 + d)n$ and*

$$C(d)\frac{n}{m}\|w\|^2 \leq \|A(J)w\|_{l_2(J)}^2 \qquad \forall w \in l_2^n,$$

where $C(d)$ is a positive constant depending only on d.

3. Proof of Theorem 1

Fix numbers $n, m \in \mathbb{N}$ and $d > 0$ satisfying $n(1 + d) \leq m$.

3.1. It suffices to prove Theorem 1 for the sets S of the form

$$S = \bigcup_{r \in I} \left[\frac{2\pi r}{m}, \frac{2\pi(r + 1)}{m}\right],$$

where $I \subset \{0, ..., m - 1\}$, $|I| = n$. Clearly, $|S| = n/m$.

3.2. Denote by
$$\mathcal{F}_I := (e^{i\frac{2\pi jr}{m}})_{r\in I, j=0,\dots,m-1}$$
the submatrix of the Fourier matrix $\mathcal{F}$ whose columns are indexed by I. Since the matrix $(\sqrt{m})^{-1}\mathcal{F}$ is orthonormal, by Corollary 1 there exists $J \subset \{0, \dots, m-1\}$, $|J| \leq (1+d)n$, such that

$$\|\mathcal{F}_I(J)w\|^2_{l_2(J)} \geq C(d)n\|w\|^2, \qquad w \in l_2(I). \tag{4}$$

3.3. Observe that every function $F \in L^2(S)$ can be written as
$$F(t) = \sum_{r\in I} F_r(t - \frac{2\pi r}{m}),$$

where $F_r \in L^2(0, \frac{2\pi}{m})$ is defined by

$$F_r(t) := F(t + \frac{2\pi r}{m})\mathbf{1}_{[0, \frac{2\pi}{m}]}(t).$$

Therefore, every function $f \in PW_S$ admits a representation

$$f(x) = \sum_{r\in I} e^{i\frac{2\pi r}{m}x} f_r(x), \qquad f_r \in PW_{[0, \frac{2\pi}{m}]},$$

where the functions $e^{i\frac{2\pi r}{m}x} f_r(x)$ are orthogonal in $L^2(\mathbb{R})$.

3.4. We now verify that the sequence

$$\Lambda := \{j + km : j \in J, k \in \mathbb{Z}\}$$

satisfies the conclusion of Theorem 1. Take any function $f \in PW_S$. Then

$$\sum_{j\in J}\sum_{k\in\mathbb{Z}} |f(j+km)|^2 = \sum_{j\in J}\sum_{k\in\mathbb{Z}} \left|\sum_{r\in I} e^{i\frac{2\pi rj}{m}} f_r(j+km)\right|^2.$$

For every $j \in J$ we apply (2) to the function $\sum_{r\in I} e^{i\frac{2\pi rj}{m}} f_r(x)$. We find that the last expression is equal to

$$\frac{1}{m}\sum_{j\in J}\int_{\mathbb{R}} \left|\sum_{r\in I} e^{i\frac{2\pi rj}{m}} f_r(x)\right|^2 dx = \frac{1}{m}\int_{\mathbb{R}} \|\mathcal{F}_I(J)(f_r(x))_{r\in I}\|^2_{l_2(J)} dx.$$

By inequality (4), we have

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 \geq C(d)\frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I} |f_r(x)|^2 dx =$$

$$C(d)\frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I} |e^{i\frac{2\pi r}{m}x} f_r(x)|^2 dx = C(d)\frac{n}{m}\int_{\mathbb{R}} |\sum_{r\in I} e^{i\frac{2\pi r}{m}x} f_r(x)|^2 dx =$$

$$C(d)\frac{n}{m}\int_{\mathbb{R}} |f(x)|^2 dx.$$

This completes the proof.

4. Proof of Theorem 2

4.1. **Auxiliary results.** We will use the following theorem of Ruzsa, [12] (see also [14] for the extension to a two-sided density).

Theorem B. *Let H be a family of finite sets of integers which is closed to translations, and such that all subsets of a set in H also belong to H. Set*

$$d(H) := \lim_{n \to \infty} \frac{\max_{A \in H} |A \cap [-n, n]|}{2n}.$$

Then there exists $\Gamma \subset \mathbb{Z}$ with $D^\sharp(\Gamma) = d(H)$ such that every finite subset of Γ belongs to H.

We note that the limit $d(H)$ always exists.

Theorem B was used by Bourgain and Tzafriri in an application of the Restricted Invertibility Theorem. Here we use it in a different way, in order to manage the case of sampling with small spectrum.

As Theorem B studies families of finite sets, it is easier applied when working with interpolating sequences (equivalently, Riesz sequences), rather than sampling sequences. A system $\{e^{i\gamma t}\}_{\gamma \in \Gamma}$ is called a Riesz sequence for $L^2(\Omega)$ if

$$A \sum_{\gamma \in \Gamma} |a_\gamma|^2 \leq \| \sum_{\gamma \in \Gamma} a_\gamma e^{i\gamma t} \|^2_{L^2(\Omega)} \qquad \forall \{a_\gamma\}_{\gamma \in \Gamma} \in l^2(\Gamma), \tag{5}$$

where A is a positive constant not depending on a_γ. The duality between Riesz and sampling sequences is well known:

Claim 1. *A sequence Λ is a sampling sequence for PW_S if and only if $\{e^{i\gamma t}\}_{\gamma \in \mathbb{Z} \setminus \Lambda}$ is a Riesz sequence for $L^2(\mathbb{T} \setminus S)$. Moreover, in this case the sampling bound in (1) and the Riesz sequence bound in (5) are equivalent, $A/2 \leq B \leq 2A$.*

Indeed, assume (1) holds. Let $P(t) = \sum_{\gamma \in \mathbb{Z} \setminus \Lambda} a_\gamma e^{i\gamma t}$. Then

$$\|P\|^2_{L^2(S)} \leq \frac{1}{B} \sum_{\Lambda} |\langle P, e^{i\lambda t} \rangle_{L^2(S)}|^2 =$$

$$\frac{1}{B} \sum_{\Lambda} |\langle P, e^{i\lambda t} \rangle_{L^2(\mathbb{T} \setminus S)}|^2 \leq \frac{1}{B} \|P\|^2_{L^2(\mathbb{T} \setminus S)},$$

Since $B \leq 1$, this proves (5) with $A \geq B/2$, $\Gamma = \mathbb{Z} \setminus \Lambda$ and $\Omega = \mathbb{T} \setminus S$. The proof of the opposite direction follows in much the same way.

4.2. **Proof of Theorem 2.** It suffices to show that Theorem 2 holds for open sets. Fix $d > 0$ and let S be an open set such that $(1 + d)|S| < 1$. Below we denote by $C(d)$ positive constants, which depend only on d.

Let Q_j be disjoint intervals such that $S = \cup_{j \in \mathbb{N}} Q_j$. For every $m \in \mathbb{N}$ denote $S_m := \cup_{j < m} Q_j$. Using Theorem 1, find for every m a set $\Lambda_m \subset \mathbb{Z}$, such that $D(\Lambda_m) < (1 + d)|S_m|$ and Λ_m is a sampling sequence for PW_{S_m} with sampling bound larger then $C(d)|S_m|$.

For every m choose a finite set Γ_m and an interval $I_m \subset \mathbb{Z}$ so that:

$$\Gamma_m \subset (\mathbb{Z} \setminus \Lambda_m) \cap I_m,$$

and

$$\frac{|\Gamma_m|}{|I_m|} > 1 - (1+d)|S_m|.$$

By Claim 1 the set of exponentials $\{e^{i\gamma t}\}_{\gamma \in \Gamma_m}$ is a Riesz sequence in $L^2(\mathbb{T} \setminus S_m)$ with Riesz sequence bound larger then $C(d)|S_m|$.

Let H be the minimal translation invariant family of finite sets in $\mathbb{Z}$, which contains all finite subsets of every Γ_m. For this H take $\Gamma \subset \mathbb{Z}$ according to Theorem B. Then

$$D^\sharp(\Gamma) \geq 1 - (1+d)|S|.$$

Take an arbitrary integer M. Then $\Gamma^M := \Gamma \cap [-M, M]$ belongs to H. So we can fix $m = m(M)$ such that Γ^M is a subset of some shifted Γ_m. Clearly m tends to infinity as M does. So $\{e^{i\gamma t}\}_{\gamma \in \Gamma^M}$ is a (finite) Riesz sequence in $L^2(\mathbb{T} \setminus S_m)$ with bound bigger then $C(d)|S_m|$. It follows $\{e^{i\gamma t}\}_{\gamma \in \Gamma}$ is a Riesz in $L^2(\mathbb{T} \setminus S)$ with bound bigger then $C(d)|S|$. Using claim 1 again we conclude that $\Lambda := \mathbb{Z} \setminus \Gamma$ satisfies the required conditions.

5. Upper bounds

5.1. **Good Bessel sequences.** In this section we discuss the possible upper bounds in a sampling process. A sequence $\Lambda \subset \mathbb{Z}$ is called a Bessel sequence for PW_S if there exists $B > 0$ such that

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B\|f\|^2 \qquad \forall f \in PW_S.$$

Here we show that a theorem of Lunin [9] implies the existence of good Bessel sequences: Sequences which are not too sparse on one side, but provide good Bessel bounds on the other side. Again, we consider spectra of small measure. We keep in mind the canonical example from Section 1 and show that the inequalities in Theorems 1 and 2 can be reversed.

Theorem 3. *Given a compact set S, one can find a sequence Λ of uniform density $D(\Lambda) > |S|$ such that*

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C|S|\|f\|^2 \qquad \forall f \in PW_S,$$

where C is an absolute positive constant.

Moreover, in this case the sequence Λ can be constructed as a finite union of arithmetic progressions.

Theorem 4. *Given a set S of positive measure, one can find a sequence Λ of density $D^\sharp(\Lambda) > |S|$ such that*

$$\sum |f(\lambda)|^2 \leq C|S|\|f\|^2 \qquad \forall f \in PW_S,$$

where C is an absolute positive constant.

5.2. **Lunin's theorem.** The type of problems studied in theorem A goes back to Kashin, [4], [5], who in connection with certain questions in the theory of orthogonal series proved that for every $m \times n$ matrix A of norm 1 with $n/m < \rho(\varepsilon)$, one can find a subset $J \subset \{1, ..., m\}$, of size $|J| = n$, so that the sub-matrix $A(J)$ has norm less then ε. Moreover, some estimate of $\rho(\varepsilon)$ was established. A sharp constant $\rho(\varepsilon)$ was found by Lunin [9]:

Theorem C. *Let A be an $m \times n$ with $\|A\| = 1$. Then there exists a subset $J \subset \{1, ..., m\}$ for which $|J| = n$ and*

$$\|A(J)w\|_{l_2^n}^2 \leq C\frac{n}{m}\|w\|^2 \qquad \forall w \in l_2^n,$$

where C is an absolute positive constant.

Observe that Theorem C can be proved also using the technique developed in [1].

One can *prove Theorem 3* in the same way as Theorem 1, the only difference is that one uses Theorem C instead of Corollary 1.

The *proof of Theorem 4* is pretty similar to the proof of Theorem 2, in which one has to use Theorem 3 instead of Theorem 1.

6. ADDITIONAL REMARKS AND OPEN PROBLEMS

6.1. **Restricted Invertibility Theorem.** The Restricted Invertibility Theorem (RIT) was proved by Bourgain and Tzafriri in [2] using a probabilistic approach, see also [3], [6], and [14]. Recently Spielman and Srivastava, [13], used the technique developed in [1] to give a short linear algebraic proof of the theorem. We formulate here the RIT as in [13].

Theorem D. *Fix $0 < d < 1$. For any operator $T : l_2^m \to l_2^m$ which satisfies $\|Te_i\| = 1$, where $\{e_i\}$ is the canonical basis for l_2^m, one can find a subset $J \subset \{1, ..., m\}$ of size $|J| > (1 - d)m/\|T\|^2$ which satisfies*

$$C(d) \sum_{i \in J} |a_i|^2 \leq \|\sum_{i \in J} a_i Te_i\|^2,$$

where $C(d) = (1 - \sqrt{1 - d})^2$.

In [2] Bourgain and Tzafriri applied the RIT to obtain a result regarding good Riesz sequences for general measurable sets. For an open set, this result may be improved by choosing the frequencies of the corresponding Riesz sequence to be uniformly distributed in $\mathbb{Z}$. To this end, we need the following corollary of the RIT.

Corollary 2. *Fix $0 < d < 1$. Let A be an $m \times n$ matrix which is a sub-matrix of some $m \times m$ orthonormal matrix and such that all of it's rows have equal l^2 norm. Then there exists a subset $J \subset \{1, ..., m\}$ for which $|J| \geq (1-d)n$ and*

$$C(d)\frac{n}{m}\|w\|^2 \leq \|A(J)^{\mathrm{T}} w\|_{l_2^n}^2 \qquad \forall w \in l_2(J),$$

where $C(d)$ is the constant given in Theorem D.

Indeed, without loss of generality we may assume that the columns of A are the first n columns of some $m \times m$ orthonormal matrix. Let P be the orthonormal projection on l_m^2 which is defined by replacing all but the first n coordinates of a vector by zero. Then $T = \sqrt{m/n}P$ satisfies the conditions of Theorem D, corresponding to the basis given by the rows of the orthonormal matrix, and $\|T\|^2 = m/n$. The corollary follows.

Theorem 5. *Fix $0 < d < 1$. Given an open set $\Omega \subset [0, 2\pi]$, there exists $\Gamma \subset \mathbb{Z}$ of uniform density $D(\Gamma) > (1-d)|\Omega|$ such that*

$$C(d)|\Omega| \sum_{\gamma \in \Gamma} |a_\gamma|^2 \leq \|\sum_{\gamma \in \Gamma} a_\gamma e^{i\gamma t}\|_{L^2(\Omega)}^2,$$

where $C(d)$ is the constant given in Theorem D.

This theorem can be proved using Corollary 2 in much the same way that Theorem 1 is obtained from Corollary 1. We omit the proof.

Remark 1. Using Claim 1, one can reformulate this result in terms of sampling. In particular, this implies the existence of good sampling sequences in the sense of the definition in Section 1, for sets whose measure is not too small. Moreover, by an appropriate choice of the constant d in Theorem 5, one can get in this way a weaker version of Theorem 1, with $|S|$ replaced by $|S|^2$ in the sampling bound. However, it is not clear whether Theorem 1 itself can be deduced this way. At least, the value of the constant $C(d)$ in Theorem D does not imply this.

6.2. **Open problems.**

1. It seems to be an important problem, whether Theorems 1 and 3 could be combined: Given a compact S of positive measure, is there a good exponential frame, that is a sequence $\Lambda \subset \mathbb{Z}$ such that the two sided inequality

$$A|S|\|f\|^2 \leq \sum |f(\lambda)|^2 \leq B|S|\|f\|^2 \qquad \forall f \in PW_S$$

   is fulfilled with absolute constants A and B? It is mentioned in [1] that the finite dimensional result of this type would imply the Kadison-Singer conjecture.

2. Another related problem: Given a compact $\Omega \subset [-\pi, \pi]$ of positive measure, is there a sequence $\Gamma \subset \mathbb{Z}$ without arbitrarily long gaps, such that $\{e^{i\gamma t}\}_{\gamma \in \Gamma}$ is a Riesz sequence in $L^2(\Omega)$? See [8], where it is proved that this question is equivalent to the Kadison-Singer conjecture for exponential frames.

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School of Mathematical Sciences, Tel Aviv University, Ramat-Aviv 69978, Israel
nitzansi@post.tau.ac.il

School of Mathematical Sciences, Tel Aviv University, Ramat-Aviv 69978, Israel
olevskii@math.tau.ac.il

University of Stavanger, Stavanger 4036, Norway
alexander.ulanovskii@uis.no